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Alternative integral representations for the Green function of the theory of ship wave resistance

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SUMMARY

Three alternative single-integral representations for the Green function of the theory of ship wave resistance are derived in a unified manner from a basic double-integral representation. These alternative single-integral representations, which essentially are modifications of well-known double-integral representations due to Michell, Havelock, and Peters, are compared and discussed. Another object of this study is to examine the field equation and the boundary condition satisfied by the Green function in the limiting case when the singular point is exactly at the free surface.

1. Introduction

For several decades, theoretical investigations of the wave-making resistance of ships were almost exclusively restricted to the evaluation and testing of Michell's celebrated wave resistance formula (mainly for idealized hulls of simple geometrical shape), and only since the advent of fast electronic computers has it become possible to contemplate the implementation of more sophisticated theories than Michell's first-order thin-ship approximation. Prediction of flow about an arbitrary ship hull – even if one greatly simplifies the real problem by neglecting any effect due to the ship propeller, the viscosity of water, the formation of spray at the ship bow, and free surface nonlinearities – presents formidable difficulties, however, from both the standpoints of the development of a satisfactory mathematical model and of the numerical implementation of any such theory.

The numerical difficulties of the problem can essentially be traced to the complexity of the mathematical expression for the Green function, or more precisely of any of the several available alternative expressions for this fundamental function. As is discussed in Appendix 1 of Eggers, Sharma, and Ward [1], there appear to be three basic alternative expressions for the Green function. The best-known of these probably is that given by Havelock [2], or a modification of it due to Lunde [3] and quoted in Wehausen and Laitone [4]. Of the two other above-mentioned alternative expressions, one is due to Peters [5], while the other may be traced back to Michell's famous paper [6], in which it is implicitly contained. Each of these three basic alternative expressions (see [1] for instance) involves one double (two-fold) integral as well as one (or two, in the case of Michell's expression) single (one-fold) integral. The double integrals represent non-oscillatory nearfield (local) flow disturbances, while the single integrals represent oscillatory disturbances in which may be recognized the classical pattern of ship waves.

With the advent of electronic computers opening up the feasibility of numerical calculations, the need for expressions for the Green function more amenable to numerical evaluation than the above-mentioned basic expressions has led to a renewal of interest in the Green function in the last ten to fifteen years, as is attested to by several studies in which various modified expressions for the Green function are proposed. A brief review of these studies may be found in Noblesse [7]. In particular, the Havelock-Lunde form [2, 3, 1] of the Green function has been the object of several investigations, in which modifications of this expression more suitable for numerical evaluation are sought. The last study in this series of investigations (see [7] for a brief review) seeking to simplify the Havelock form of the Green function is that by Shen and Farell [8], where an expression essentially identical to the expression defined by formulas (12) in the present study was obtained. This study of Shen and Farell and a study by Andersson [9] prompted the present author [7] to modify the Peters form [5, 1] of the Green function in a manner analogous to Shen and Farell's modification of the Havelock expression. Specifically, the modified Havelock and Peters expression obtained in [8] and [7] and given in this study by expressions (12) and (7), respectively, mainly differ from the original expressions obtained by Havelock [2] and Peters [5] and given in [1] in that the double integrals in these basic expressions (which represent nonoscillatory near-field flow disturbances as was noted previously) have been transformed into single integrals involving the exponential integral $E_1(\zeta)$, which can be regarded as a standard function for analytical as well as numerical purposes, in the integrands.

One might suspect that the third of the above-mentioned alternative basic forms for the Green function, namely the expression given implicitly by Michell [6] and explicitly in [1], can be modified in a manner analogous to the modifications of the Havelock and Peters expressions obtained in [8] and [7]. This modified Michell form of the Green function in fact is obtained in the present study, and is specifically given by formulas (9). Like expressions (12) and (7) for the modified Havelock and Peters representations for the Green function, expressions (9) for the modified Michell representation differ from the basic Michell expression given in [1] in that the double integral representing a nonoscillatory near-field disturbance is expressed as a single integral in terms of the exponential integral $E_1(\zeta)$ in the integrand.

Another object of the present study (in addition to obtaining the above-mentioned modified Michell form of the Green function) is to show that this modified Michell expression and the modified Havelock and Peters expressions, obtained previously in [8] and [7], can be derived in a unified straightforward manner from the basic double-integral representation given by formula (5), specifically by evaluating the inner (complex) integral by means of contour integration. This derivation shows that the origin of the three alternative single-integral representations given by formulas (7), (9), and (12) can be traced to interchanging the order of integration with respect to the Fourier-transform variables ξ and η in the double integral (5) and to transforming this double integral from an integral in the 'Cartesian space' ξ, η to an integral in the 'polar space' ρ, θ (by performing the change of variables $\xi = \rho\cos\theta, \eta = \rho\sin\theta$).

Among the three alternative single-integral representations given by formulas (7), (9), and (12), the modified Peters representation (7) appears to be the most desirable for the purpose of numerical calculations. Expression (7a) for the near-field disturbance can be expressed in the form given by equation (13), as was shown previously in Noblesse [10], which is well suited for numerical evaluation. The corresponding expressions for the gradient of the Green function are given in [10]. Another appealing feature of the modified Peters integral representation (7) is that expression (7a) for the near-field disturbance is well-suited for the purpose of obtaining an ascending series useful for small values of $|\mathbf{x}'|$. As a matter of fact, the two algebraic terms on the right side of equation (13) are the first two terms in this ascending series, and the third term in the series is given in [10], equation (5).

However, expression (7a) is not suited for obtaining an asymptotic expansion for large values of $|\mathbf{x}'|$. A complementary integral representation for the near-field disturbance in the Peters representation (7) therefore is obtained in this study. This complementary integral representation is given by equation (14), which can be obtained by combining the modified Peters and Havelock representations (7) and (12). The integral representation (14) is suited for the purpose of obtaining an asymptotic expansion useful for evaluating the near-field disturbance for large and moderate values of $|\mathbf{x}'|$. As a matter of fact, an asymptotic expansion for the near-field disturbance has previously been obtained from expression (14) in Noblesse [11] in the particular case when $y' \equiv 0$, for which the last integral on the right side vanishes.

A last and main object of this study is to show that the Green function of the theory of ship wave resistance satisfies the field equations and free surface boundary conditions given by equations (17) and (18). The equations corresponding to the case $c < 0$ or $z < 0$ in equations (17) or (18), respectively, are well known. However, the equations corresponding to the limiting case $c = 0$ or $z = 0$ do not appear to have been explicitly indicated previously, to the author's knowledge, except in Noblesse [12] where the limiting form of these equations obtained by putting $\epsilon = 0$ in formulas (17) and (18) was derived on the basis of a simple physical argument. This physical reasoning is completed here by a more formal mathematical demonstration. Equations (18) are important for obtaining an integral equation for the velocity potential of the flow caused by steady rectilinear motion of a ship in a quiescent sea, as is shown in [12].

2. Formulation of the problem

The linearized free surface boundary condition appropriate to a free surface gravity flow observed from a system of coordinates in translation with constant speed U along the X axis in the positive direction takes the familiar form

$$[g\partial_Z + (U\partial_X - \partial_T)^2]\Phi = 0 \text{ on } Z = 0,$$

where g represents the acceleration of gravity, Z is the vertical coordinate with the free surface taken as the plane $Z = 0$, T is the time, Φ is the velocity potential of the flow, and the symbols ∂_Z, ∂_X, ∂_T mean differentiation with respect to Z, X, and T, respectively. It is convenient to define nondimensional variables in terms of U, U/g, U^2/g, and U^3/g as characteristic velocity, time, length, and velocity potential, respectively; we thus have

$$t = Tg/U, \mathbf{x} = \mathbf{X}g/U^2, \phi = \Phi g/U^3$$

as nondimensional time, coordinates, and potential, respectively, in terms of which the free surface condition takes the form

$$[\partial_z + (\partial_x - \partial_t)^2]\phi = 0 \quad \text{on } z = 0.$$

In this study, we are interested in flows that do not vary with time. As is well known, and is discussed for instance in Stoker [13], steady-state free surface gravity-flow problems require the use of a 'radiation condition' for complete determinacy. A convenient alternative approach, used for instance in Lighthill [14], consists in defining a steady-state flow as the limit – as the small positive auxiliary parameter ϵ vanishes – of a time-dependent flow defined by a potential of the form $\phi(\mathbf{x}, t) = \varphi(\mathbf{x})\exp(\epsilon t)$, so that one is faced with a traditional 'initial-value problem' (with the obvious 'initial conditions' $\phi = 0$ and $\phi_t = 0$ for $t = -\infty$). The free surface condition for the 'spatial component' $\varphi(\mathbf{x})$ of the potential $\varphi(\mathbf{x})\exp(\epsilon t)$ then takes the form

$$[\partial_z + (\partial_x - \epsilon)^2]\varphi = 0 \text{ on } z = 0,$$

or equivalently

$$\varphi_z + \varphi_{xx} - 2\epsilon\varphi_x + \epsilon^2\varphi = 0 \text{ on } z = 0.$$

It may be worthwhile to note here in passing that if the term $\epsilon^2\varphi$ is neglected – as can be justified for $\epsilon \ll 1$ – this free surface condition becomes identical with the often-used condition obtained by invoking Rayleigh's 'artificial viscosity' concept, as is described for instance in Lamb [15].

The Green function $G(\mathbf{x}; \mathbf{a}, \epsilon)$ associated with the above free surface boundary condition is the solution of the problem defined by

$$\nabla^2 G = \delta(x - a)\delta(y - b)\delta(z - c) \quad \text{in } z < 0, \tag{1a}$$

$$[\partial_z + (\partial_x - \epsilon)^2]G = 0 \quad \text{on } z = 0, \tag{1b}$$

$$G \to 0 \quad \text{as } |\mathbf{x} - \mathbf{a}| \to \infty, \tag{1c}$$

where $\delta(\ )$ represents the usual Dirac 'δ function', $\mathbf{a}$ (a, b, c) is the position vector of the singularity (unit source), and c is assumed to be strictly negative. The Green function of the theory of ship wave resistance is the function $G(\mathbf{x}; \mathbf{a}) \equiv G(\mathbf{x}; \mathbf{a}, \epsilon = +0)$.

3. Double-integral representation of the Green function

A particular solution of the Poisson equation (1a) is given by $4\pi G = -1/r$ where $r \equiv |\mathbf{x} - \mathbf{a}|$ is the distance between the 'source point' $\mathbf{a}$ and the 'field point' $\mathbf{x}$, as is well known and can readily be verified. The general solution of equation (1a) can thus be written as

$$4\pi G(\mathbf{x}; \mathbf{a}, \epsilon) = -1/r + H(\mathbf{x}; \mathbf{a}, \epsilon), \tag{2}$$

where the function H is regular harmonic in the lower half space $z < 0$, and evidently is to be determined from the boundary conditions (1b, c). Indeed substitution of expression (2) into equations (1a, b, c) yields

$$\nabla^2 H = 0 \quad \text{in } z < 0, \tag{3a}$$

$$[\partial_z + (\partial_x - \epsilon)^2] H = [\partial_z + (\partial_x - \epsilon)^2](1/r) \quad \text{on } z = 0, \tag{3b}$$

$$H \to 0 \quad \text{as } r \to \infty. \tag{3c}$$

The above problem can be solved by using a double Fourier transform with respect to the horizontal coordinates x and y. The double Fourier transform of the function $H(\mathbf{x}; \mathbf{a}, \epsilon)$ is denoted by $H^{**}(\xi, \eta, z; \mathbf{a}, \epsilon)$ and defined as

$$H^{**}(\xi, \eta, z; \mathbf{a}, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, e^{i(\xi x + \eta y)} H(\mathbf{x}; \mathbf{a}, \epsilon).$$

The corresponding Fourier transform of the function $1/r$ is

$$(1/r)^{**} = (1/\rho)\exp[-\rho\, |z - c| + i(\xi a + \eta b)]$$

where $\rho \equiv (\xi^2 + \eta^2)^{1/2}$, as may be verified. By taking the double Fourier transform with respect to x and y of equations (3a, b, c) we may then obtain the following 'Fourier-transformed problem' for the function $H^{**}(z; \xi, \eta, \mathbf{a}, \epsilon)$:

$$d^2 H^{**}/dz^2 - \rho^2 H^{**} = 0 \quad \text{in } z < 0, \tag{4a}$$

$$dH^{**}/dz - (\xi - i\epsilon)^2 H^{**} = -[1 + (\xi - i\epsilon)^2/\rho] e^{\rho c + i(\xi a + \eta b)} \quad \text{on } z = 0, \tag{4b}$$

$$H^{**} \to 0 \quad \text{as } z \to -\infty. \tag{4c}$$

The general solution of the ordinary differential equation (4a) is $H^{**} = A \exp(\rho z) + B \exp(-\rho z)$, where A and B are arbitrary constants. The boundary condition (4c) shows that $B = 0$, while the constant A then can be determined from the free surface condition (4b). We thus may obtain

$$H^{**} = \frac{1}{\rho} e^{\rho(z+c) + i(\xi a + \eta b)} - \frac{2}{\rho - (\xi - i\epsilon)^2} e^{\rho(z+c) + i(\xi a + \eta b)}.$$

The function $H(\mathbf{x}; \mathbf{a}, \epsilon)$ may now be obtained by taking the inverse double Fourier transform of the function H^{**}, namely

$$H(x, y, z; \mathbf{a}, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi\, e^{-i(\xi x + \eta y)} H^{**}(\xi, \eta, z; \mathbf{a}, \epsilon).$$

Upon substituting the above expression for H^{**}, we may obtain

$$H(\mathbf{x}; \mathbf{a}, \epsilon) = \frac{1}{r'} - \frac{1}{\pi} \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi\, \frac{e^{\rho z' - i(\xi x' + \eta y')}}{\rho - (\xi - i\epsilon)^2},$$

where $x' \equiv x - a, y' \equiv y - b, z' \equiv z + c$, and $r' \equiv (x'^2 + y'^2 + z'^2)^{1/2}$ by definition.

By substituting the above expression for H into equation (2), we may finally obtain

$$4\pi G(\mathbf{x};\mathbf{a},\epsilon) = -\frac{1}{r} + \frac{1}{r'} - \frac{1}{\pi}\int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi \, \frac{e^{z'(\xi^2+\eta^2)^{1/2} - i(x'\xi+y'\eta)}}{(\xi^2+\eta^2)^{1/2} - (\xi - i\epsilon)^2}. \tag{5}$$

In the following three sections, three alternative expressions – each one involving single (one-fold) integrals only – will be derived from the classical basic double-integral representation (5).

4. The Cartesian x – y integration and the resulting Peters single-integral representation

Let the double-integral representation (5) for the Green function $G(\mathbf{x};\mathbf{a},\epsilon)$ be expressed in the form

$$4\pi G = -\frac{1}{r} + \frac{1}{r'} - \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-iy'\eta}\, I_1(\eta; x', z', \epsilon)\, d\eta, \tag{6}$$

where the inner integral I_1 is defined as

$$I_1(\eta; x', z', \epsilon) = \int_{-\infty}^{\infty} \frac{e^{z'(\xi^2+\eta^2)^{1/2} - ix'\xi}}{(\xi^2+\eta^2)^{1/2} - (\xi - i\epsilon)^2}\, d\xi. \tag{6a}$$

Evaluation of the integral I_1 by treating it as an integral along the real axis in the integration contours in the complex ξ plane shown in Figure 1 yields* in the limit $\epsilon = +0$

$$I_1 = i\int_{-\infty}^{\infty} \frac{e^{-|x'|(\mu^2+\eta^2)^{1/2} + iz'\mu}}{(\mu^2+\eta^2+i\mu)(\mu^2+\eta^2)^{1/2}}\, \mu d\mu - \frac{H(-x')4\pi}{2\alpha - 1/\alpha}\, e^{z'(\alpha^2+\eta^2)^{1/2}} \sin(x'\alpha), \tag{6b}$$

where $\alpha = \{[1 + (1 + 4\eta^2)^{1/2}]/2\}^{1/2}$ and $H(\ )$ is the usual Heaviside 'unit-step function'.

By substituting expression (6b) for the inner integral I_1 into equation (6), and after performing some transformations,* we may express the Green function $G(\mathbf{x};\mathbf{a})$ of the theory of ship wave resistance in the form

$$4\pi G(\mathbf{x};\mathbf{a}) = -1/r + N_1(\mathbf{x}') + W_1(\mathbf{x}'), \tag{7}$$

where the functions $N_1(\mathbf{x}')$ and $W_1(\mathbf{x}')$ are defined as

$$N_1(\mathbf{x}') = \frac{1}{r'} + \frac{2}{\pi}\int_{-1}^{1} Im\, e^{\zeta_1} E_1(\zeta_1)\, dt, \tag{7a}$$

in which $r' = |\mathbf{x}'|$ as was defined previously and ζ_1 is the complex function given by

$$\zeta_1 = [z'(1-t^2)^{1/2} + y't + i|x'|](1-t^2)^{1/2}, \tag{7a'}$$

* For details, see Appendix 1.

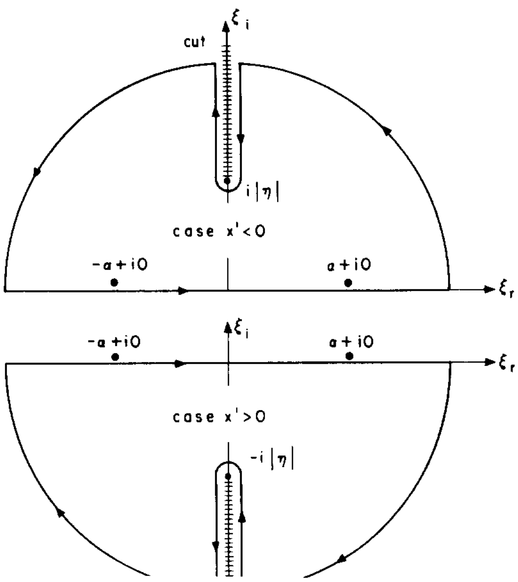

Figure 1. Integration contours in the complex $\xi = \xi_r + i\xi_i$ plane for $x' < 0$ (above) and $x' > 0$ (below).

and

$$W_1(x') = H(-x')4\int_{-\infty}^{\infty} Im\, e^{z'(1+t^2)+i(x'+y't)(1+t^2)^{1/2}}\, dt. \tag{7b}$$

In formula (7a), as indeed hereafter in this study, the function $E_1(\zeta)$ is the exponential integral defined by

$$E_1(\zeta) = \int_{\zeta}^{\infty} \frac{e^{-\lambda}}{\lambda}\, d\lambda,$$

where the path of integration is assumed to exclude the origin and not to cross the negative real axis in the complex λ plane, in accordance with the definition used in Abramowitz and Stegun [16], p. 228.

5. The Cartesian $y - x$ integration and the resulting Michell single-integral representation

By interchanging the order of integration in the double integral in equation (5), we may express the Green function $G(\mathbf{x}; \mathbf{a}, \epsilon)$ in the form

$$4\pi G = -\frac{1}{r} + \frac{1}{r'} - \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-ix'\xi} I_2(\xi; y', z', \epsilon) d\xi, \tag{8}$$

where the inner integral I_2 is defined as

$$I_2(\xi; y', z', \epsilon) = \int_{-\infty}^{\infty} \frac{e^{z'(\eta^2+\xi^2)^{1/2} - i|y'|\eta}}{(\eta^2+\xi^2)^{1/2} - (\xi - i\epsilon)^2}\, d\eta, \tag{8a}$$

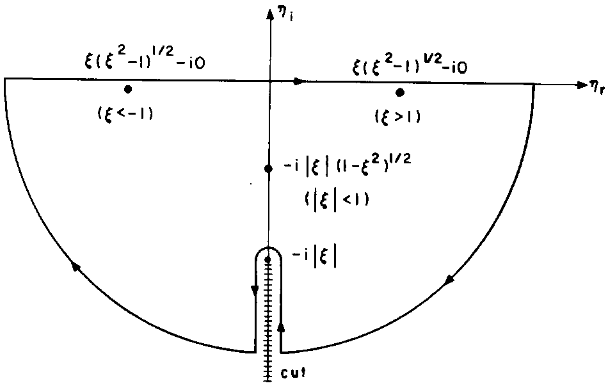

Figure 2. Integration contour in the complex $\eta = \eta_r + i\eta_i$ plane

in which the obvious symmetry with respect to y' has been used explicitly.

Evaluation of the integral I_2 by treating it as an integral along the real axis in the integration contour in the complex η plane shown in Figure 2 yields* in the limit $\epsilon = +0$

$$I_2 = i\int_{-\infty}^{\infty} \frac{e^{-|y'|(\mu^2+\xi^2)^{1/2}+iz'\mu}}{(i\mu-\xi^2)(\mu^2+\xi^2)^{1/2}}\,\mu d\mu - 2\pi i R, \tag{8b}$$

where R is the residue given by

$$R = \begin{cases} \exp[z'\xi^2 - i|y'|\xi(\xi^2-1)^{1/2}]\,\xi/(\xi^2-1)^{1/2} & \text{for } |\xi|>1 \\ i\exp[z'\xi^2 - |y'|\,|\xi|(1-\xi^2)^{1/2}]\,|\xi|/(1-\xi^2)^{1/2} & \text{for } |\xi|<1 \end{cases} \tag{8b'}$$

By substituting expression (8b) for the inner integral I_2 into equation (8), and after performing some transformations*, we may express the Green function $G(\mathbf{x};\mathbf{a})$ of the theory of ship wave resistance in the form

$$4\pi G(\mathbf{x};\mathbf{a}) = -1/r + N_2(\mathbf{x}') + W_2(\mathbf{x}'), \tag{9}$$

where the functions $N_2(\mathbf{x}')$ and $W_2(\mathbf{x}')$ are defined as

$$N_2(\mathbf{x}') = \frac{1}{r'} + \frac{2}{\pi}\int_{-\infty}^{\infty} Im\, e^{\zeta_2}E_1(\zeta_2)\,\frac{t\,dt}{(1+t^2)^{1/2}}, \tag{9a}$$

in which ζ_2 is the complex function given by

$$\zeta_2 = [x' - z't + i\,|y'|(1+t^2)^{1/2}]\,t, \tag{9a'}$$

* For details, see Appendix 2.

and

$$W_2(\mathbf{x}') = 4\int_0^\infty Im\, e^{z'(1+t^2)+i(x'+|y'|t)(1+t^2)^{1/2}}\,dt$$

$$-4\int_0^1 Re\, e^{z'(1-t^2)-|y'|t(1-t^2)^{1/2}+ix'(1-t^2)^{1/2}}\,dt. \tag{9b}$$

6. The polar $\rho - \theta$ integration and the resulting Havelock single-integral representation

By expressing the Cartesian variables ξ and η in the double integral in equation (5) in terms of the polar variables ρ and θ, as follows: $\xi = \rho\cos\theta$ and $\eta = \rho\sin\theta$, we may express the Green function $G(\mathbf{x};\mathbf{a},\epsilon)$ in the form

$$4\pi G = \frac{-1}{r} + \frac{1}{r'} - \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \int_0^\infty \rho d\rho\, \frac{e^{[z'-i(x'\cos\theta+|y'|\sin\theta)]\rho}}{\rho-(\rho\cos\theta-i\epsilon)^2}. \tag{10}$$

By performing some simple transformations (including the change of variable $t = \tan\theta$) we may obtain* in the limit $\epsilon = +0$

$$4\pi G = \frac{-1}{r} + \frac{1}{r'} + \frac{2}{\pi}\int_{-\infty}^{\infty} Re\, I_3(t;\mathbf{x}')dt, \tag{11}$$

where the inner integral I_3 is defined as

$$I_3(t;\mathbf{x}') = \int_0^\infty \frac{e^{[z'-i(x'+|y'|t)(1+t^2)^{-1/2}]\rho}}{\rho-(1+t^2+i0)}\,d\rho. \tag{11a}$$

Evaluation of the integral I_3 by treating it as an integral along the positive real axis in the integration contours in the complex ρ plane shown in Figure 3 yields*

$$I_3(t;\mathbf{x}') = e^{\zeta}E_1(\zeta) + H(-x'/|y'|-t)2\pi i e^{\zeta}, \tag{11b}$$

where ζ is the complex function given by

$$\zeta = z'(1+t^2) - i(x'+|y'|t)(1+t^2)^{1/2}. \tag{11b'}$$

By substituting expression (11b) into equation (11), and after performing some elementary transformations, we may obtain

$$4\pi G(\mathbf{x};\mathbf{a}) = -1/r + N_3(\mathbf{x}') + W_3(\mathbf{x}'), \tag{12}$$

* For details, see Appendix 3.

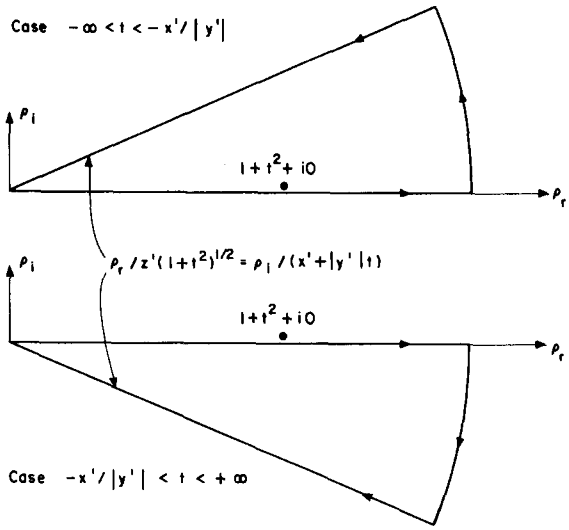

Figure 3. Integration contours in the complex $\rho = \rho_r + i\rho_i$ plane for $-\infty < t < -x'/|y'|$ (above) and $-x'/|y'| < t < +\infty$ (below).

where the functions $N_3(\mathbf{x}')$ and $W_3(\mathbf{x}')$ are defined as

$$N_3(\mathbf{x}') = \frac{1}{r'} + \frac{2}{\pi}\int_{-\infty}^{\infty} Re\, e^{\zeta_3} E_1(\zeta_3)dt, \tag{12a}$$

in which ζ_3 is the complex function given by

$$\zeta_3 = [z'(1+t^2)^{1/2} + i(x'+y't)]\,(1+t^2)^{1/2}, \tag{12a'}$$

and

$$W_3(\mathbf{x}') = 4\int_{x'/|y'|}^{\infty} Im\, e^{z'(1+t^2)+i(x'-|y'|t)(1+t^2)^{1/2}}\,dt. \tag{12b}$$

7. Comparison and discussion of above alternative expressions for the Green function

The three above-derived alternative expressions (7), (9), and (12) for the Green function are all written as the sum of three terms, as follows:

$$4\pi G(\mathbf{x};\mathbf{a}) = -1/r + N(\mathbf{x}') + W(\mathbf{x}').$$

In this expression, the term $-1/r$, where $r \equiv |\mathbf{x} - \mathbf{a}|$ is the distance between the 'source point' $\mathbf{a}$ and the 'field point' $\mathbf{x}$, is the usual Green function for potential flow in an unbounded fluid (that is, in the absence of a free surface), while the effect of the free surface is accounted for by the terms $N(\mathbf{x}')$ and $W(\mathbf{x}')$, where $\mathbf{x}'(x' \equiv x - a, y' \equiv y - b, z' \equiv z + c)$ is the vector joining the mirror image of the 'source point' $\mathbf{a}$ with respect to the free surface $z = 0$ to the 'field point' $\mathbf{x}$; the terms $W(\mathbf{x}')$ in expressions (7), (9), and (12), which stem from the residues in the complex integrations depicted in Figures 1, 2, and 3, are oscillatory functions of x' and y' representing the wave pattern caused by the moving source, while the terms $N(\mathbf{x}')$ are nonoscillatory functions representing near-field (local) disturbances.

There are obvious similarities among the mathematical expressions for the 'near-field' and 'wave' disturbances $N(\mathbf{x}')$ and $W(\mathbf{x}')$ given by equations (7), (9), and (12). In particular, expressions (7a), (9a), and (12a) for the 'near-field disturbances' all involve a single integral with fixed limits of integration, namely $[-1, +1]$ for expression (7a) and $[-\infty, +\infty]$ for expressions (9a) and (12a), in terms of the exponential integral $E_1(\zeta)$ in the integrand. The function $E_1(\zeta)$ can be regarded as a standard function for all purposes, analytical as well as numerical. Indeed, the function $E_1(\zeta)$ possesses both a convergent ascending series for $|\zeta| < \infty$ and an asymptotic expansion as $|\zeta| \to \infty$, which are given for instance by equations (5.1.11) and (5.1.51) on pages 229 and 231 in Abramowitz and Stegun [16]. Furthermore, for the purpose of numerical evaluation these series can be supplemented by an intermediate approximation, so-called 'approximation by equivalent poles', due to Hershey [17]. An interesting common feature of expressions (7a), (9a), and (12a) for $N(\mathbf{x}')$ is that they are even functions of both y' (as was evidently to be expected on physical grounds) and x', as can easily be verified. Indeed, the evenness of expressions (7a) and (9a) with respect to x' and y', respectively, is self evident from formulas (7a′) and (9a′), and the evenness of expressions (7a) and (9a) with respect to y' and x', respectively, and of expression (12a) with respect to x' and y', can be established by performing the change of variable $\tau = -t$ in the integrals (7a), (9a), and (12a). Expressions (7a), (9a), and (12a) for $N(\mathbf{x}')$ thus represent (near-field) disturbances that are 'symmetric' upstream and downstream for the moving source.

Differences among the alternative mathematical expressions (7), (9), and (12) for the Green function are most readily apparent from expressions (7b), (9b), and (12b) for the 'wave disturbances' $W(\mathbf{x}')$. In particular, expression (7b) for $W_1(\mathbf{x}')$ involves a single integral, while two integrals are involved in expression (9b) for $W_2(\mathbf{x}')$. A common feature of the integrals in expressions (7b) and (9b) is that the limits of integration are fixed, whereas the lower limit of integration in expression (12b) for $W_3(\mathbf{x}')$ is $x'/|y'|$, which is obviously not constant but varies with the relative positions of the 'source' and 'field' points; this feature in fact appears to be a major drawback of the 'polar representation' (12), as is indeed discussed in [1]. From the physical point of view, an appealing feature of expression (7b) for the 'wave disturbance' $W_1(\mathbf{x}')$ resides in the fact that this expression is identically zero for $x' > 0$, that is upstream from the source, so that this expression makes particularly vivid the fact that the wave pattern created by a moving source is behind the source, as is well known. From this physical point of view, as well as from the point of view of convenience for purposes of mathematical analysis and of numerical calculations, expression (7b) for the 'wave disturbance' would seem to be the most desirable of the three expressions (7b), (9b), and (12b).

It may be also of interest to note that the Cartesian $x - y$ representation (7) is closest in form to the expression for the two-dimensional Green function, corresponding to a two-dimensional line source of unit strength, which can be written as

$$2\pi G(\mathbf{x};\mathbf{a}) = \ell n\, r + N(\mathbf{x}') + W(\mathbf{x}'),$$

where $r \equiv |\mathbf{x} - \mathbf{a}| = [(x-a)^2 + (z-c)^2]^{1/2}$, $\mathbf{x}'(x' \equiv x - a, z' \equiv z + c)$, and the 'near-field' and 'wave' disturbances $N(\mathbf{x}')$ and $W(\mathbf{x}')$ are given by

$$N(\mathbf{x}') = \ln r' + 2 Re\, e^{z' + ix'} E_1(z' + ix'), \text{ with } r' \equiv (x'^2 + z'^2)^{1/2},$$

$$W(\mathbf{x}') = H(-x') 2\pi\, Im\, e^{z' + ix'}.$$

Another interesting feature of the modified Peters integral representation (7) resides in the property that expression (7a) for the near-field disturbance $N(\mathbf{x}')$ is well-suited for the purpose of obtaining an ascending series useful for evaluating $N(\mathbf{x}')$ for small values of $|\mathbf{x}'|$. As a matter of fact, it is shown in [10] that it is computationally advantageous to use the following modified form of expression (7a):

$$N_1(\mathbf{x}') = \frac{1}{r'} - 2\left(1 + \frac{-z'}{r' + |x'|}\right) + \frac{2}{\pi} \int_{-1}^{1} Im\, [e^{\zeta_1} E_1(\zeta_1) + \ln \zeta_1 + \gamma] dt, \tag{13}$$

where the two algebraic terms on the right side actually are the first two terms in the above-mentioned ascending series of $N_1(\mathbf{x}')$ about the origin $\mathbf{x}' = 0$, ζ_1 is the complex function defined by equation (7a′), and $\gamma = 0.577...$ is Euler's constant. The third term in this ascending series is also given in [10], equation (5), and other terms in the series can in principle be obtained without difficulty.

However, a drawback of the modified Peters integral representation (7) is that expression (7a) for the near-field disturbance is not suited for obtaining an asymptotic expansion for large values of $|\mathbf{x}'|$. The modified Havelock expression (12) may be useful for this purpose, as is shown below. By comparing expressions (7) and (12), we may express the near-field disturbance $N_1(\mathbf{x}')$ in expression (7) in the form $N_1 = N_3 + W_3 - W_1$. Use of expressions (7b) and (12b) yields

$$W_3 - W_1 = 4 \int_{|x'|/|y'|}^{\infty} Im\, e^{z'(1+t^2) + i[|x'| - |y'| t (1+t^2)^{1/2}]} dt,$$

as may be obtained after some simple algebraic manipulations. Furthermore, it is convenient to express equation (12a) in the equivalent form

$$N_3(\mathbf{x}') = -\frac{1}{r'} + \frac{2}{\pi} \int_{-\infty}^{\infty} Re \left[e^{\zeta_3} E_1(\zeta_3) - \frac{1}{\zeta_3} \right] dt,$$

where the algebraic term, $-1/r'$, on the right side is the first term in the asymptotic expansion of $N_3(\mathbf{x}')$ as $|\mathbf{x}'| \to \infty$.

We may finally obtain the following alternative expression for the near-field disturbance $N_1(\mathbf{x}')$ in equation (7):

$$N_1(\mathbf{x}') = -\frac{1}{r'} + \frac{2}{\pi} \int_{-\infty}^{\infty} Re \left[e^{\zeta'} E_1(\zeta') - \frac{1}{\zeta'} \right] dt + 4 \int_{|x'|/|y'|}^{\infty} Im\, e^{\zeta'} dt, \tag{14}$$

where ζ' is the complex function given by

$$\zeta' = [z'(1+t^2)^{1/2} + i(|x'| - |y'|t)]\,(1+t^2)^{1/2}. \tag{14a}$$

Expression (14) is suited for obtaining an asymptotic expansion useful for evaluating $N_1(\mathbf{x}')$ for large and moderate values of $|\mathbf{x}'|$. As a matter of fact, an asymptotic expansion for $N_1(\mathbf{x}')$ has previously been obtained from expression (14) in [11] in the particular case when $y' \equiv 0$, for which the last integral on the right side vanishes.

8. On the field equation and boundary condition satisfied by the Green function

The Green function $G(\mathbf{x}\,;\mathbf{a}\,,\epsilon = +0)$ may be interpreted physically as the linearized velocity potential, at the 'field point' $\mathbf{x}$, of the flow due to a unit source (that is, a point source where fluid is produced at a flow rate equal to unity), located at point $\mathbf{a}$, in steady rectilinear motion at a constant depth ($-c > 0$) below the free surface of an otherwise quiescent sea, with the position vectors $\mathbf{x}$ and $\mathbf{a}$ referred to a translating system of coordinates moving with the source. This physical interpretation of the Green function is well known, and indeed can readily be justified from equations (1). In the limiting case $c = 0$, however, the unit source evidently is no longer fully submerged, so that the above physical interpretation becomes somewhat ambiguous. A more natural physical interpretation for this limiting case is to assume that the unit outflow produced at point $(a, b, c = 0)$ now stems from a flux across the (undisturbed) free surface $z = 0$. The mathematical implications of this complementary interpretation, for the case $c = 0$, are examined below.

With respect to a system of coordinates in translation with constant speed U along the X axis in the positive direction, as was defined previously, the linearized dynamic free surface boundary condition can be expressed as

$$gE + (\partial_T - U\partial_X)\,\Phi = 0 \qquad \text{on } Z = 0,$$

where the pressure was taken as zero at the free surface, and $E(X, Y, T)$ represents the elevation of the free surface above or below the mean water level $Z = 0$. In the presence of a fluid flux, $Q(X, Y, T)$ say, across the free surface, the linearized kinematic free surface boundary condition takes the form

$$\Phi_Z = (\partial_T - U\partial_X)\,E - Q \qquad \text{on } Z = 0,$$

where $Q < 0$ evidently means that fluid is sucked away across the free surface. Elimination of the free surface elevation E between the above dynamic and kinematic free surface conditions then yields

$$[g\partial_Z + (U\partial_X - \partial_T)^2]\,\Phi = -gQ \qquad \text{on } Z = 0,$$

which becomes

$$[\partial_z + (\partial_x - \partial_t)^2]\,\phi = -q \qquad \text{on } z = 0$$

in terms of the nondimensional variables $t = Tg/U$, $\mathbf{x} = \mathbf{X}g/U^2$, $\phi = \Phi g/U^3$ (used previously), and $q = Q/U$. If – as throughout the present study – we restrict our attention to flows with an exponential time-dependence factor of the form $\phi(\mathbf{x}, t) = \varphi(\mathbf{x})\exp(\epsilon t)$ and $q(x, y, t) = \sigma(x, y)$ $\exp(\epsilon t)$, the above free surface boundary condition finally takes the form

$$[\partial_z + (\partial_x - \epsilon)^2]\,\varphi = -\sigma \quad \text{on } z = 0.$$

The expressions for the free surface boundary condition previously given in Section 2 evidently correspond to assuming Q, q, or σ is zero in the above expressions.

A unit flux across the free surface at point $(a, b, c = 0)$ corresponds to $\sigma(x, y) = \delta(x - a)\,\delta(y - b)$, so that the previously-mentioned physical interpretation of the Green function $G(\mathbf{x}; \mathbf{a}, \epsilon)$ in the limiting case $c = 0$ suggests that this 'limit Green function', say $G_\ell(\mathbf{x}; a, b, \epsilon) \equiv G(\mathbf{x}; a, b, c = 0, \epsilon)$, must satisfy the following equations

$$\nabla^2 G_\ell = 0 \quad \text{in } z < 0, \tag{15a}$$

$$[\partial_z + (\partial_x - \epsilon)^2]\,G_\ell = -\delta(x - a)\delta(y - b) \quad \text{on } z = 0, \tag{15b}$$

$$G_\ell \to 0 \quad \text{as } |\mathbf{x} - \bar{\mathbf{a}}| \to \infty, \tag{15c}$$

which are to be compared to equations (1a, b, c) satisfied by $G(\mathbf{x}; \mathbf{a}, \epsilon)$ if $c < 0$. A mathematical demonstration of the above physically motivated equations can readily be provided by verifying that the solution $G_\ell(\mathbf{x}; a, b, \epsilon)$ of the problem defined by equations (15) actually is identical with the 'limit Green function' obtained by replacing c by zero in formula (5), namely

$$4\pi G_\ell(\mathbf{x}; a, b, \epsilon) = \frac{-1}{\pi} \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi\, \frac{e^{z(\xi^2 + \eta^2)^{1/2} - i(x'\xi + y'\eta)}}{(\xi^2 + \eta^2)^{1/2} - (\xi - i\epsilon)^2}, \tag{16}$$

or conversely by verifying that the 'limit Green function' G_ℓ given by formula (16) does in fact satisfy equations (15)*.

It may thus be seen, in summary, that the Green function $G(\mathbf{x}; \mathbf{a}, \epsilon)$ of ship wave resistance theory (where the limit $\epsilon = +0$ ultimately is implied) satisfies the following equations

$$\left.\begin{aligned} &\nabla^2 G = \delta(x - a)\delta(y - b)\delta(z - c) \quad \text{in } z < 0 \\ &[\partial_z + (\partial_x - \epsilon)^2]\,G = 0 \quad \text{on } z = 0 \end{aligned}\right\} \text{if } c < 0$$

$$\left.\begin{aligned} &\nabla^2 G = 0 \quad \text{in } z < 0 \\ &[\partial_z + (\partial_x - \epsilon)^2]\,G = -\delta(x - a)\delta(y - b) \quad \text{on } z = 0 \end{aligned}\right\} \text{if } c = 0 \tag{17}$$

* These mathematical demonstrations may be found in Appendix 4.

It may readily be seen, for instance from expression (5), that the Green function $G(\mathbf{x};\mathbf{a},\epsilon)$ actually is a function of the four variables $x' \equiv x - a, y' \equiv y - b, z' \equiv z + c$, and $(z - c)^2$, so that this function is invariant under the changes of variables $x \longleftrightarrow -a, y \longleftrightarrow -b, z \longleftrightarrow c$. By performing these changes of variables in equations (17), we may then see that the Green function $G(\mathbf{x};\mathbf{a},\epsilon)$ must also satisfy the following equations:

$$\left.\begin{array}{ll} \nabla_a^2 G = \delta(x-a)\delta(y-b)\delta(z-c) & \text{in } c<0 \\ [\partial_c + (\partial_a+\epsilon)^2]\, G = 0 & \text{on } c=0 \end{array}\right\} \text{if } z<0$$

$$\left.\begin{array}{ll} \nabla_a{}^2 G = 0 & \text{in } c<0 \\ [\partial_c + (\partial_a+\epsilon)^2]\, G = -\delta(x-a)\delta(y-b) & \text{on } c=0 \end{array}\right\} \text{if } z=0 \tag{18}$$

where ∇_a represents the differential operator $(\partial_a, \partial_b, \partial_c)$. These equations, with $\epsilon = 0$, have been derived previously in [12], where they are used for the purpose of obtaining an integral equation for the velocity potential of the flow caused by steady rectilinear motion of a ship in an otherwise quiescent sea.

9. Conclusion

In summary, the Green function $G(\mathbf{x};\mathbf{a})$ in the theory of ship wave resistance may be expressed in the form

$$4\pi G(\mathbf{x};\mathbf{a}) = -1/r + N(\mathbf{x}') + W(\mathbf{x}'). \tag{19}$$

In this expression, the term $-1/r$, where $r \equiv |\mathbf{x} - \mathbf{a}|$ is the distance between the 'singular point' $\mathbf{a}$ and the 'field point' $\mathbf{x}$, is the Green function for potential flow in an unbounded fluid, and the terms $N(\mathbf{x}')$ and $W(\mathbf{x}')$, where $\mathbf{x}'(x' \equiv x - a, y' \equiv y - b, z' \equiv z + c)$ is the vector joining the mirror image of $\mathbf{a}$ with respect to the plane of the free surface $z = 0$ to the field point $\mathbf{x}$, account for the effect of the free surface. The term $W(\mathbf{x}')$ represents the wave pattern following the singularity, and is given by the integral

$$W(\mathbf{x}') = H(-x')\, 4 \int_{-\infty}^{\infty} Im\, e^{z'(1+t^2)+i(x'+y't)(1+t^2)^{1/2}}\, dt. \tag{20}$$

The term $N(\mathbf{x}')$, on the other hand, represents a nonoscillatory near-field (local) disturbance.
This near-field disturbance is given by the integral

$$N(\mathbf{x}') = \frac{1}{r'} + \frac{2}{\pi} \int_{-1}^{1} Im\, e^{\zeta} E_1(\zeta) dt, \tag{21}$$

where $r' \equiv (x'^2 + y'^2 + z'^2)^{1/2}$, $\zeta \equiv [z'(1-t^2)^{1/2} + y't + i\,|x'|]\,(1-t^2)^{1/2}$ and E_1 is the usual exponential integral. For purposes of numerical calculations, a convenient alternative expression is

$$N(\mathbf{x}') = \frac{1}{r'} - 2\left(1 + \frac{-z'}{r' + |x'|}\right) + \frac{2}{\pi}\int_{-1}^{1} Im\,[e^{\zeta}E_1(\zeta) + \ell n\zeta + \gamma]\,dt, \tag{21a}$$

where $\gamma = 0.577...$ is Euler's constant. Expression (21) is well suited for obtaining an ascending series about the origin $\mathbf{x}' = 0$. Indeed, the first term in this series is $1/r'$, so that we have

$$N(\mathbf{x}') \sim 1/r' \text{ as } r' \to 0,$$

and the second term is shown in expression (21a). The third term in the series is given in [10], where the expression for ∇N corresponding to expressions (21) and (21a) can also be found.

By using the relation $1/r' = -(1/\pi)\int_{-1}^{1} Im\,(1/\zeta)dt$, the following alternative expression for the near-field disturbance $N(\mathbf{x}')$:

$$N(\mathbf{x}') = -\frac{1}{r'} + \frac{2}{\pi}\int_{-1}^{1} Im\left(e^{\zeta}E_1(\zeta) - \frac{1}{\zeta}\right)dt \tag{21b}$$

can be obtained from expression (21). Although expression (21b) may be useful for some analytical purposes, it is not well suited for purposes of numerical evaluation. Neither is expression (21b) suited for obtaining an asymptotic expansion for large values of r'. A complementary integral representation suited for that purpose is

$$N(\mathbf{x}') = -\frac{1}{r'} + \frac{2}{\pi}\int_{-\infty}^{\infty} Re\left[e^{\zeta'}E_1(\zeta') - \frac{1}{\zeta'}\right]dt + 4\int_{|x'|/|y'|}^{\infty} Im\, e^{\zeta'}dt, \tag{21'}$$

where $\zeta' \equiv [z'\,(1+t^2)^{1/2} + i(|x'| - |y'|t)]\,(1+t^2)^{1/2}$. The first term in the asymptotic expansion of $N(\mathbf{x}')$ indeed is given by

$$N(\mathbf{x}') \sim -1/r' \text{ as } r' \to \infty.$$

In the particular case when $y' \equiv 0$, the last integral on the right side of expression (21′) vanishes, and an asymptotic expansion for $N(x', 0, z')$ as $x'^2 + z'^2 \to \infty$ has been obtained in [11] from the resulting simplified expression.

Finally, it was shown that the Green function $G(\mathbf{x}; \mathbf{a})$ satisfies the following equations:

$$\left.\begin{aligned} &\nabla_a^2 G = \delta(x-a)\,\delta(y-b)\,\delta(z-c) \quad \text{in } c<0 \\ &\partial G/\partial c + \partial^2 G/\partial a^2 = 0 \quad \text{on } c=0 \end{aligned}\right\} \text{ if } z<0 \tag{22a}$$

$$\left.\begin{aligned} &\nabla_a^2 G = 0 \quad \text{in } c<0 \\ &\partial G/\partial c + \partial^2 G/\partial a^2 = -\delta(x-a)\,\delta(y-b) \quad \text{on } c=0 \end{aligned}\right\} \text{ if } z=0. \tag{22b}$$

These equations are used in [12] for obtaining an integral equation for the velocity potential of the flow due to steady rectilinear motion of a ship in a quiescent sea.

Appendix 1: The Cartesian $x-y$ integration

The integrand of the inner integral I_1 defined by formula (6a) has two simple poles given by the solutions of the equation $(\xi^2+\eta^2)^{1/2}=(\xi-i\epsilon)^2$, which may be shown to be $\pm\alpha+i0$ in the limit $\epsilon=+0$, where α is defined as $\alpha=\{[1+(1+4\eta^2)^{1/2}]/2\}^{1/2}$. These poles are depicted in Figure 1. The integral I_1 can be treated as a complex integral along the real axis in the complex ξ plane. This path of integration can be modified in the usual manner by introducing a properly-chosen closed integration contour comprising the real axis. Any such integration contour must evidently satisfy the condition $Re\,[z'(\xi^2+\eta^2)^{1/2}-ix'\xi]\leqslant 0$ as $|\xi|\to\infty$, where Re denotes the real part. It may be verified that this condition is satisfied for the integration contours shown in Figure 1, where the upper and lower contours correspond to the cases $x'<0$ and $x'>0$, respectively.

Let us consider the case $x'>0$ (lower contour) first. It can be verified that we have $(\xi^2+\eta^2)^{1/2}=\mp i(\xi_i^2-\eta^2)^{1/2}$ for $\xi=\pm 0+i\xi_i$ on the two sides of the cut $\xi_r=0$, $-\infty<\xi_i<-|\eta|$. We may then obtain

$$I_1=\int_{-\infty}^{-|\eta|}\frac{e^{x'\xi_i+iz'(\xi_i^2-\eta^2)^{1/2}}}{\xi_i^2+i(\xi_i^2-\eta^2)^{1/2}}\,id\xi_i+\int_{-|\eta|}^{-\infty}\frac{e^{x'\xi_i-iz'(\xi_i^2-\eta^2)^{1/2}}}{\xi_i^2-i(\xi_i^2-\eta^2)^{1/2}}\,id\xi_i.$$

The changes of variable $\mu=(\xi_i^2-\eta^2)^{1/2}$ and $\mu=-(\xi_i^2-\eta^2)^{1/2}$ in the first and second integral, respectively, finally yield

$$I_1=i\int_{-\infty}^{\infty}\frac{e^{-x'(\mu^2+\eta^2)^{1/2}+iz'\mu}}{(\mu^2+\eta^2+i\mu)(\mu^2+\eta^2)^{1/2}}\,\mu d\mu.$$

We now consider the case $x'<0$ (upper contour). It can be verified that we have $(\xi^2+\eta^2)^{1/2}=\pm i(\xi_i^2-\eta^2)^{1/2}$ for $\xi=\pm 0+i\xi_i$ on the two sides of the cut $\xi_r=0$, $|\eta|<\xi_i<\infty$. We may then obtain

$$I_1=\int_{+\infty}^{|\eta|}\frac{e^{x'\xi_i-iz'(\xi_i^2-\eta^2)^{1/2}}}{\xi_i^2-i(\xi_i^2-\eta^2)^{1/2}}\,id\xi_i+\int_{|\eta|}^{+\infty}\frac{e^{x'\xi_i+iz'(\xi_i^2-\eta^2)^{1/2}}}{\xi_i^2+i(\xi_i^2-\eta^2)^{1/2}}\,id\xi_i+R,$$

where R stems from the residues at the poles $\pm\alpha+i0$; specifically we have $R=2\pi i\,[Res(\alpha)+Res(-\alpha)]$, in which the residue $Res(\pm\alpha)$ at $\pm\alpha$ can be shown to be given by

$$Res(\pm\alpha)=\mp\exp[z'(\alpha^2+\eta^2)^{1/2}\mp ix'\alpha]/(2\alpha-1/\alpha).$$

By substituting this result into the above expression for I_1, and by performing the changes of variables $\mu=-(\xi_i^2-\eta^2)^{1/2}$ and $\mu=(\xi_i^2-\eta^2)^{1/2}$ in the first and second integral, respectively, we can finally obtain

$$I_1=i\int_{-\infty}^{\infty}\frac{e^{x'(\mu^2+\eta^2)^{1/2}+iz'\mu}}{(\mu^2+\eta^2+i\mu)(\mu^2+\eta^2)^{1/2}}\,\mu d\mu-\frac{4\pi}{2\alpha-1/\alpha}\,e^{z'(\alpha^2+\eta^2)^{1/2}}\sin(x'\alpha).$$

Comparison of the above expressions for the inner integral I_1 in the cases $x' > 0$ and $x' < 0$ then readily yields formula (6b). By substituting expression (6b) into equation (6), we may then express the Green function G in the form of formula (7), with the functions $N_1(\vec{x}')$ and $W_1(\mathrm{x}')$ given by

$$N_1(\mathrm{x}') = \frac{1}{r'} - \frac{i}{\pi}\int_{-\infty}^{\infty} d\eta e^{-iy'\eta} \int_{-\infty}^{\infty} \frac{e^{-|x'|(\mu^2+\eta^2)^{1/2}+iz'\mu}}{(\mu^2+\eta^2+i\mu)(\mu^2+\eta^2)^{1/2}}\,\mu d\mu,$$

$$W_1(\mathrm{x}') = H(-x')\,4\int_{-\infty}^{\infty} \frac{e^{z'(\alpha^2+\eta^2)^{1/2}-iy'\eta}}{2\alpha - 1/\alpha}\,\sin(x'\alpha)d\eta.$$

The change of variable $\eta = t\,(1+t^2)^{1/2}$ in the above expression for the integral W_1 may be shown to give

$$W_1(\mathrm{x}') = H(-x')\,4\int_{-\infty}^{\infty} e^{z'(1+t^2)-iy't(1+t^2)^{1/2}} \sin[x'(1+t^2)^{1/2}]dt,$$

from which expression (7b) can then be obtained.

We now consider the double integral N_1, which may readily be written in the form

$$N_1(\mathrm{x}') = \frac{1}{r'} - \frac{i}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-|x'|(\mu^2+\eta^2)^{1/2}+i(y'\eta+z'\mu)}}{(\mu^2+\eta^2+i\mu)(\mu^2+\eta^2)^{1/2}}\,\mu d\eta d\mu.$$

By performing the change of variables $\eta = \rho\cos\theta$ and $\mu = \rho\sin\theta$, that is by changing from the Cartesian coordinates (η, μ) to the polar coordinates (ρ, θ), we may obtain

$$N_1(\mathrm{x}') = \frac{1}{r'} - \frac{i}{\pi}\int_{-\pi}^{\pi} I(\theta;\mathrm{x}')\sin\theta\, d\theta,$$

where the inner integral $I(\theta;\mathrm{x}')$ is given by

$$I(\theta;\mathrm{x}') = \int_0^{\infty} \frac{e^{-[|x'|-i(y'\cos\theta+z'\sin\theta)]\rho}}{\rho + i\sin\theta}\,d\rho.$$

The change of variable $\tau = \rho + i\sin\theta$ yields

$$I(\theta;\mathrm{x}') = e^{\zeta}\int_{i\sin\theta}^{\infty} e^{-[|x'|-i(y'\cos\theta+z'\sin\theta)]\tau}\,\frac{d\tau}{\tau},$$

where ζ is the complex function defined as

$$\zeta = (y'\cos\theta + z'\sin\theta + i\,|x'|)\sin\theta.$$

By performing the change of variable $\lambda = [|x'| - i(y'\cos\theta + z'\sin\theta)]\tau$, we may then obtain

$$I(\theta;\mathrm{x}') = e^{\zeta}\int_{\zeta}^{\infty} e^{-\lambda}\,\frac{d\lambda}{\lambda} = e^{\zeta}E_1(\zeta),$$

where $E_1(\zeta)$ is the usual exponential integral, in which the path of integration excludes the origin and does not cross the negative real axis in the complex λ plane.

By substituting this expression for $I(\theta; \mathrm{x}')$ into the last of the above expressions for $N_1(\mathrm{x}')$, we then have

$$N_1(\mathrm{x}') = \frac{1}{r'} - \frac{i}{\pi}\int_{-\pi}^{\pi} e^{\zeta}E_1(\zeta)\sin\theta d\theta$$

$$= \frac{1}{r'} - \frac{i}{\pi}\left\{\int_0^{\pi} e^{\zeta}E_1(\zeta)\sin\theta d\theta + \int_{-\pi}^{0} e^{\zeta}E_1(\zeta)\sin\theta d\theta\right\}$$

$$= \frac{1}{r'} - \frac{i}{\pi}\left\{\int_0^{\pi} e^{\zeta}E_1(\zeta)\sin\theta d\theta - \int_0^{\pi} e^{\bar{\zeta}}E_1(\bar{\zeta})\sin\psi d\psi\right\},$$

where the change of variable $\psi = \theta + \pi$ was performed, and $\bar{\zeta}$ is the complex conjugate of the above-defined complex function ζ (with θ replaced by ψ). By using the relation $\exp(\bar{\zeta})E_1(\bar{\zeta}) = \overline{\exp(\zeta)E_1(\zeta)}$ we may then obtain

$$N_1(\mathrm{x}') = \frac{1}{r'} - \frac{i}{\pi}\int_0^{\pi} [e^{\zeta}E_1(\zeta) - \overline{e^{\zeta}E_1(\zeta)}]\sin\theta d\theta =$$

$$= \frac{1}{r'} + \frac{2}{\pi}\int_0^{\pi} Im\, e^{\zeta}E_1(\zeta)\sin\theta d\theta,$$

from which formula (7a) can finally be obtained by performing the change of variable $t = \cos\theta$.

Appendix 2: The Cartesian $y - x$ integration

By multiplying the numerator and denominator of the integrand of the inner integral $I_2(\xi; y', z', \epsilon)$ defined by equation (8a) by the expression $(\eta^2 + \xi^2)^{1/2} + (\xi - i\epsilon)^2$, and by rearranging the denominator, we may express the integral I_2 in the form

$$I_2 = \int_{-\infty}^{\infty} \frac{e^{z'(\eta^2+\xi^2)^{1/2} - i|y'|\eta}\,[(\eta^2 + \xi^2)^{1/2} + (\xi - i\epsilon)^2]}{[\eta + \{(\xi - i\epsilon)^4 - \xi^2\}^{1/2}]\,[\eta - \{(\xi - i\epsilon)^4 - \xi^2\}^{1/2}]}\, d\eta.$$

For $\epsilon \ll 1$, we have $(\xi - i\epsilon)^4 - \xi^2 \sim \xi^2(\xi^2 - 1) - i4\epsilon\xi^3$. It can then be shown that the poles $\pm [(\xi - i\epsilon)^4 - \xi^2]^{1/2}$ of the integrand of the above integral are given by $\pm [\xi(\xi^2 - 1)^{1/2} - i0]$ for $|\xi| > 1$ and $\pm [0 - i\xi(1 - \xi^2)^{1/2}]$ for $|\xi| < 1$, in the limit $\epsilon = +0$. The integral I_2 can be treated as a complex integral along the real axis in the complex η plane. This path of integration may be modified in the usual manner by introducing a properly-chosen closed integration contour comprising the real axis. Such an integration contour must evidently satisfy the condition $Re[z'(\eta^2 + \xi^2)^{1/2} - i|y'|\eta] \leqslant 0$ as $|\eta| \to \infty$. It may be verified that this condition is satisfied for the integration contour shown in Figure 2.

It can be shown that we have $(\eta^2 + \xi^2)^{1/2} = \mp i(\eta_i^2 - \xi^2)^{1/2}$ for $\eta = \pm 0 + i\eta_i$ on the two sides of the cut $\eta_r = 0$, $-\infty < \eta_i < -|\xi|$. In the limit $\epsilon = +0$, the integral I_2 defined by equation (8a) can then be expressed in the form

$$I_2 = \int_{-\infty}^{-|\xi|} \frac{e^{|y'|\eta_i + iz'(\eta_i^2 - \xi^2)^{1/2}}}{i(\eta_i^2 - \xi^2)^{1/2} - \xi^2} id\eta_i + \int_{-|\xi|}^{-\infty} \frac{e^{|y'|\eta_i - iz'(\eta_i^2 - \xi^2)^{1/2}}}{-i(\eta_i^2 - \xi^2)^{1/2} - \xi^2} id\eta_i - 2\pi iR.$$

In the above equation R is the residue at the pole $\xi(\xi^2 - 1)^{1/2} - i0$, for $|\xi| > 1$, or $-i|\xi|(1 - \xi^2)^{1/2}$, for $|\xi| < 1$, given by expressions (8b′). Formula (8b) can then be obtained from the above expression for I_2 by performing the changes of variables $\mu = (\eta_i^2 - \xi^2)^{1/2}$ and $\mu = -(\eta_i^2 - \xi^2)^{1/2}$ in the first and second integral, respectively.

By substituting expression (8b) into equation (8), we may then express the Green function G in the form of formula (9), with the functions $N_2(\mathbf{x}')$ and $W_2(\mathbf{x}')$ defined as

$$N_2(\mathbf{x}') = \frac{1}{r'} - \frac{i}{\pi}\int_{-\infty}^{\infty} d\xi e^{-ix'\xi} \int_{-\infty}^{\infty} \frac{e^{-|y'|(\mu^2 + \xi^2)^{1/2} + iz'\mu}}{(i\mu - \xi^2)(\mu^2 + \xi^2)^{1/2}} \mu d\mu,$$

$$W_2(\mathbf{x}') = 2i\int_{-\infty}^{\infty} e^{-ix'\xi} R(\xi; |y'|, z')d\xi,$$

in which $R(\xi; |y'|, z')$ is the residue given by expressions (8b′). We may then obtain

$$W_2(\mathbf{x}') = 4\int_1^{\infty} e^{z'\xi^2} \sin[|y'|\xi(\xi^2 - 1)^{1/2} + x'\xi]\xi(\xi^2 - 1)^{-1/2} d\xi$$
$$-4\int_0^1 e^{z'\xi^2 - |y'|\xi(1-\xi^2)^{1/2}} \cos(x'\xi)\xi(1 - \xi^2)^{-1/2} d\xi,$$

from which expression (9b) can finally be obtained by performing the changes of variables $\xi = (1 + t^2)^{1/2}$ and $\xi = (1 - t^2)^{1/2}$ in the first and second integral, respectively.

We now consider the double integral N_2, which may readily be written in the form

$$N_2(\mathbf{x}') = \frac{1}{r'} + \frac{i}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-[|y'|(\mu^2 + \xi^2)^{1/2} + i(x'\xi - z'\mu)]}}{(\xi^2 - i\mu)(\mu^2 + \xi^2)^{1/2}} \mu d\xi d\mu.$$

By performing the change of variables $\xi = \rho\cos\theta$ and $\mu = \rho\sin\theta$, that is by changing from the Cartesian coordinates (ξ, μ) to the polar coordinates (ρ, θ), we may obtain

$$N_2(\mathbf{x}') = \frac{1}{r'} + \frac{i}{\pi}\int_{-\pi/2}^{3\pi/2} I(\theta; \mathbf{x}') \sin\theta \sec^2\theta d\theta,$$

where the inner integral $I(\theta; \mathbf{x}')$ is given by

$$I(\theta; \mathbf{x}') = \int_0^{\infty} \frac{e^{-[|y'| + i(x'\cos\theta - z'\sin\theta)]\rho}}{\rho - i\sin\theta \sec^2\theta} d\rho.$$

The change of variable $\tau = \rho - i \sin\theta \sec^2\theta$ yields

$$I(\theta;\mathbf{x}') = e^{\zeta} \int_{-i \sin\theta \sec^2\theta}^{\infty} e^{-[|y'|+i(x'\cos\theta - z'\sin\theta)]\tau} \frac{d\tau}{\tau},$$

where ζ is the complex function defined as

$$\zeta = (x'\cos\theta - z'\sin\theta - i\,|y'|)\sin\theta\sec^2\theta.$$

By performing the change of variable $\lambda = [|y'| + i(x'\cos\theta - z'\sin\theta)]\tau$, we may then obtain

$$I(\theta;\mathbf{x}') = e^{\zeta}\int_{\zeta}^{\infty} e^{-\lambda}\frac{d\lambda}{\lambda} = e^{\zeta}E_1(\zeta),$$

where $E_1(\zeta)$ is the usual exponential integral.

By substituting this expression for $I(\theta;\mathbf{x}')$ into the last of the above expressions for $N_2(\mathbf{x}')$, we may then have

$$\begin{aligned} N_2(\mathbf{x}') &= \frac{1}{r'} + \frac{i}{\pi}\int_{-\pi/2}^{3\pi/2} e^{\zeta}E_1(\zeta)\sin\theta\sec^2\theta d\theta \\ &= \frac{1}{r'} + \frac{i}{\pi}\left\{\int_{-\pi/2}^{\pi/2} e^{\zeta}E_1(\zeta)\sin\theta\sec^2\theta d\theta + \int_{\pi/2}^{3\pi/2} e^{\zeta}E_1(\zeta)\sin\theta\sec^2\theta d\theta\right\} \\ &= \frac{1}{r'} + \frac{i}{\pi}\left\{\int_{-\pi/2}^{\pi/2} e^{\zeta}E_1(\zeta)\sin\theta\sec^2\theta d\theta - \int_{-\pi/2}^{\pi/2} e^{\bar{\zeta}}E_1(\bar{\zeta})\sin\psi\sec^2\psi d\psi\right\} \end{aligned}$$

where the change of variable $\psi = \theta - \pi$ was performed, and $\bar{\zeta}$ is the complex conjugate of the above-defined complex number ζ (with θ replaced by ψ). By using the relation $\exp(\bar{\zeta})E_1(\bar{\zeta}) = \overline{\exp(\zeta)E_1(\zeta)}$ we may then obtain

$$\begin{aligned} N_2(\mathbf{x}') &= \frac{1}{r'} + \frac{i}{\pi}\int_{-\pi/2}^{\pi/2} [e^{\zeta}E_1(\zeta) - \overline{e^{\zeta}E_1(\zeta)}]\sin\theta\sec^2\theta d\theta \\ &= \frac{1}{r'} + \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} Im\,\overline{e^{\zeta}E_1(\zeta)}\sin\theta\sec^2\theta d\theta \\ &= \frac{1}{r'} + \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} Im\,e^{\bar{\zeta}}E_1(\bar{\zeta})\sin\theta\sec^2\theta d\theta \end{aligned}$$

from which formula (9a) can finally be obtained by performing the change of variable $t = \tan\theta$.

Appendix 3: The polar $\rho - \theta$ integration

For $\epsilon \ll 1$, we have $\rho - (\rho\cos\theta - i\epsilon)^2 \sim \rho - \rho^2\cos^2\theta + 2i\epsilon\rho\cos\theta = -\rho\cos^2\theta\,(\rho - \sec^2\theta - 2i\epsilon\sec\theta)$, so that formula (10) may be expressed in the form

$$4\pi G = \frac{-1}{r} + \frac{1}{r'} + \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} I(\theta; \mathbf{x}', \epsilon)\sec^2\theta d\theta,$$

with the inner integral $I(\theta; \mathbf{x}', \epsilon)$ defined as

$$I(\theta; \mathbf{x}', \epsilon) = \int_0^\infty \frac{e^{[z' - i(x'\cos\theta + |y'|\sin\theta)]\rho}}{\rho - (\sec^2\theta + 2i\epsilon\sec\theta)}\, d\rho.$$

We may write

$$4\pi G = \frac{-1}{r} + \frac{1}{r'} + \frac{1}{\pi}\left\{\int_{-\pi/2}^{\pi/2} I(\theta; \mathbf{x}', \epsilon)\sec^2\theta d\theta + \int_{\pi/2}^{3\pi/2} I(\theta; \mathbf{x}', \epsilon)\sec^2\theta d\theta\right\}$$

$$= \frac{-1}{r} + \frac{1}{r'} + \frac{1}{\pi}\left\{\int_{-\pi/2}^{\pi/2} I(\theta; \mathbf{x}', \epsilon)\sec^2\theta d\theta + \int_{-\pi/2}^{\pi/2} \bar{I}(\psi; \mathbf{x}', \epsilon)\sec^2\psi d\psi\right\},$$

where the change of variable $\psi = \theta - \pi$ was performed in the last integral, and $\bar{I}$ denotes the complex conjugate of I. We thus have

$$4\pi G = \frac{-1}{r} + \frac{1}{r'} + \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} Re\, I(\theta; \mathbf{x}'', \epsilon)\sec^2\theta d\theta,$$

from which expressions (11) and (11a) can then be derived by performing the change of variable $t = \tan\theta$.

The integral $I_3(t, \mathbf{x}')$ defined by formula (11a) may be treated as a complex integral along the positive real axis in the complex $\rho \equiv \rho_r + i\rho_i$ plane. This path of integration can be modified in the usual manner by introducing a properly-chosen closed integration contour comprising the positive real axis. Such an integration contour must evidently satisfy the condition $Re[z' - i(x' + |y'|t)(1 + t^2)^{-1/2}]\,\rho \leq 0$ as $|\rho| \to \infty$. It may be verified that this condition is satisfied for the integration contours shown in Figure 3, where the upper and lower contours correspond to the cases $-\infty < t < -x'/|y'|$ and $-x'/|y'| < t < +\infty$, respectively. The characteristic feature of these integration contours is that we have $Im[z' - i(x' + |y'|t)(1 + t^2)^{-1/2}]\,\rho = 0$ along the radial lines defined by the equation $\rho_r/z'(1 + t^2)^{1/2} = \rho_i/(x' + |y'|t)$. A convenient alternative form for this equation is $\rho = \rho_r + i\rho_i = [-z' - i(x' + |y'|t)(1 + t^2)^{-1/2}]\,\sigma$, where $\sigma \geq 0$ is a parameter, so that we have $[z' - i(x' + |y'|t)(1 + t^2)^{-1/2}]\rho = -[z'^2 + (x' + |y'|t)^2/(1 + t^2)]\,\sigma$ along these radial lines.

The integrand of the integral $I_3(t; \mathbf{x}')$ defined by formula (11a) has a simple pole at $\rho = 1 + t^2 + i0$, which is inside the integration contour in the case $-\infty < t < -x'/|y'|$, and outside in the case $-x'/|y'| < t < +\infty$, as is shown in Figure 3. We may then obtain

$$I_3 = \int_0^\infty \frac{e^{-[z'^2 + (x' + |y'|t)^2/(1+t^2)]\sigma}}{[z' + i(x' + |y'|t)(1+t^2)^{-1/2}]\,\sigma + 1 + t^2}\,[z' + i(x' + |y'|t)(1+t^2)^{-1/2}]d\sigma$$

$$+ H(-x'/|y'| - t)2\pi i\, Res(1 + t^2),$$

where $H(\ )$ is the usual Heaviside 'unit-step' function, and $Res(1 + t^2)$ is the residue at the pole $\rho = 1 + t^2$. It may readily be seen from formula (11a) that we have

$$Res(1 + t^2) = \exp(\zeta),$$

where ζ is the complex function defined by equation (11b′). Multiplication of the numerator and denominator of the integrand of the integral in the above expression for I_3 by the expression $z' - i(x' + |y'|t)(1 + t^2)^{-1/2}$, followed by the change of variable $\tau = [z'^2 + (x' + |y'|t)^2/(1 + t^2)]\,\sigma$, then yields

$$I_3 = \int_0^\infty \frac{e^{-\tau}}{\tau + \zeta}\,d\tau + H(-x'/|y'| - t)2\pi i e^{\zeta}.$$

By performing the change of variable $\lambda = \tau + \zeta$, we can now obtain

$$I_3 = e^{\zeta}\int_\zeta^\infty \frac{e^{-\lambda}}{\lambda}\,d\lambda + H(-x'/|y'| - t)2\pi i e^{\zeta},$$

from which expression (11b) can finally be obtained.

By substituting expression (11b) into equation (11), we may then express the Green function G in the form of formula (12), with the functions $N_3(\mathbf{x}')$ and $W_3(\mathbf{x}')$ defined as

$$N_3(\mathbf{x}') = \frac{1}{r'} + \frac{2}{\pi}\int_{-\infty}^{\infty} Re\, e^{\zeta}E_1(\zeta)dt,$$

$$W_3(\mathbf{x}') = 4\int_{-\infty}^{-x'/|y'|} Re\, i\, e^{\zeta}dt = 4\int_{-\infty}^{-x'/|y'|} Im\, e^{\bar{\zeta}}\,dt,$$

where $\bar{\zeta}$ is the complex conjugate of the complex function ζ. Formula (12b) can finally be obtained by performing the change of variable $\tau = -t$ in the above expression for $W_3(\mathbf{x}')$. The relations $Re \exp(\zeta)E_1(\zeta) = Re\, \overline{\exp(\zeta)E_1(\zeta)}$ and $\overline{\exp(\zeta)E_1(\zeta)} = \exp(\bar{\zeta})E_1(\bar{\zeta})$ yield

$$N_3(\mathbf{x}') = \frac{1}{r'} + \frac{2}{\pi}\int_{-\infty}^{\infty} Re\, e^{\bar{\zeta}}\,E_1(\bar{\zeta})dt,$$

from which formulas (12a) and (12a′) can finally be obtained by noting that $|y'|$ can be replaced by y', as is obvious if $y' > 0$, while for $y' < 0$ we have $|y'| = -y'$, which however becomes y' by performing the change of variable $\tau = -t$ in the integral (12a).

Appendix 4: Free surface condition satisfied by the Green function

The problem defined by equations (15a, b, c) may be solved in the same manner as was used previously for solving the problem defined by equations (3a, b, c), namely by using a double Fourier transform with respect to the horizontal coordinates x and y. We may then obtain the 'Fourier-transformed problem'

$$d^2 G_\ell^{**}/dz^2 - (\xi^2 + \eta^2) G_\ell^{**} = 0 \quad \text{in } z < 0,$$

$$dG_\ell^{**}/dz - (\xi - i\epsilon)^2 G_\ell^{**} = -\exp[i(\xi a + \eta b)]/2\pi \quad \text{on } z = 0,$$

$$G_\ell^{**} \to 0 \quad \text{as } z \to -\infty,$$

where G_ℓ^{**} is the double-Fourier transform of G_ℓ. The solution of the above 'Fourier-transformed problem' may readily be found to be

$$G_\ell^{**} = \frac{-1}{2\pi} \frac{e^{(\xi^2+\eta^2)^{1/2} z + i(\xi a + \eta b)}}{(\xi^2+\eta^2)^{1/2} - (\xi - i\epsilon)^2},$$

from which expression (16) may finally be obtained by taking the inverse double Fourier transform.

Conversely, it may be shown that the 'limit Green function' G_ℓ given by expression (16) satisfies equations (15a, b, c). Equations (15a) and (15c) can readily be checked. As for the free surface condition (15b), we have

$$[\partial_z + (\partial_x - \epsilon)^2] G_\ell = -\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix'\xi} d\xi\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iy'\eta} d\eta\right) \quad \text{on } z = 0,$$

from which we may obtain

$$[\partial_z + (\partial_x - \epsilon)^2] G_\ell = -\delta(x')\delta(y') = -\delta(x-a)\delta(y-b) \quad \text{on } z = 0$$

by virtue of the relations

$$1 = \int_{-\infty}^{\infty} e^{i\xi x}\delta(x)dx, \ \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi} d\xi,$$

expressing the (well-known) fact that $\delta(x)$ and 1 are Fourier transform pairs.

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